[1955-60]

To be published in the

British Journal for the Philosophy of Science

PROBLEM SOLVING

by

Michael Polanyi

1. General heuristics

There is a purposive tension from which no fully awake animal is free. It consists in a readiness to perceive and to act, or more generally speaking, to make sense of its own situation, both intellectually and practically. From these routine efforts to retain control of itself and of its surroundings, we can see emerging a process of problem solving, when the effort tends to fall into two stages, a first stage of perplexity, followed by a second stage of doing and perceiving which dispels this perplexity. We may say that the animal has seen a problem if its perplexity lasts for some time and we can clearly recognise that it tries to find a solution to the situation which puzzles it. In doing so the animal is searching for a hidden aspect of the situation, the existence of which it surmises and for the finding or achieving of which the manifest features of the situation serve it as tentative clues or instruments.

To see a problem is a definite addition to knowledge, as much as it is to see a tree or to see a mathematical proof or a joke. It is a surmise which can be true or false, depending on whether the hidden possibilities of which it assumes the existence do actually exist or not. To recognise a problem which can be solved and is worth solving is in fact a discovery in its own right. Famous mathematical problems have descended from generation to generation, leaving in their wake a long trail of achievements stimulated by the attempt at solving them. Accordingly, at the level of animal experiments, we see the psychologist demonstrating to the animal the presence of a problem in order to start it off in search of a solution. A rat in a discrimination box is made to realise that there is food hidden in one of two compartments, both of which are accessible by pushing open its door. Only if he has grasped this will he start searching for a sign which discriminates the door with food behind it from that of the empty compartment. Similarly, animals will not start solving a maze unless they are made aware of the fact that there exists a path through it, with some reward at its outlet. In Köhler's 'insight' experiments his chimpanzees grasped their problem from the start and marked their appreciation of the task by composing themselves quietly to concentrate on it.

Accident usually plays some part in discovery and its part may be predominant. Learning experiments can be so arranged that, in the absence of any definitely understood problem, discovery can only be accidental.[1] Mechanistically minded psychologists who devise such experiments would explain all learning as the lucky outcome of random behaviour. This conception of learning underlies also the cyberneticist model of a machine which 'learns' by selecting a 'habit' which has proved successful in a series of random trials. I shall disregard this model of heuristics and continue to explore the process of discovery resulting from intelligent effort irrespective of the neural model that may be proposed for it.

Intelligent problem-solving is manifested among animals most dramatically in Köhler's experiments on chimpanzees, whose behaviour already presents the characteristic stages through which, according to Poincaré, discovery is achieved in mathematics. I have already mentioned the first: the appreciation of a problem. A chimpanzee in a cage within sight of a bunch of bananas out of its reach, neither makes any futile

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1. Guthrie and Horton placed a cat in a cage in which a small pole placed in the midst of the floor acted as release mechanism. Cats who had touched the pole by accident and found themselves freed in consequence, quickly realised the connection and proceeded to repeat their releasing action in an exactly stereotype manner. The situation in which the cat was placed offered no intelligible problem to the cat and the solution, found accidentally, showed no clear understanding of the release mechanism; the role played by intelligence in the whole process was negligible. (Comp. Hilgard, Theories of Learning, New York, 1948, p.68).

effort to get hold of it by sheer force, nor abandons its desire of acquiring the prize, but settles down instead to an unusual calm, while its eyes survey the situation all round the target; it has recognised the situation as problematical and is searching for a solution.[1] We may acknowledge this (using the terminology of Wallas based on Poincaré) as the stage of Preparation.[2]

In the most striking cases of 'insight' observed by Köhler, this preparatory stage is suddenly followed by intelligent action. Sharply breaking its calm, the animal proceeds to carry out a stratagem by which it secures its aim, or at least shows that it has grasped a principle by which this can be done. Its unhesitating manner suggests that it is guided by a clear conception of its proposed operation. This conception is its discovery, or at least - since it may not always prove practicable - its tentative discovery. We may recognise in its coming the stage of Illumination.

The practical realisation of the principle discovered by insight often presents difficulties, which may even prove unsurmountable. The manipulations by which the animal puts his insight to the test of practical realisation may be regarded as the stage of Verification.

Actually, Poincaré observed four stages of discovery: Preparation, Incubation, Illumination, Verification. But the second of these, Incubation, can be observed only in a rudimentary form in chimpanzees. Yet the observation described in some detail by Köhler in which one of his animals sustained its effort by solving a problem even while otherwise occupied for a while[3] anticipates to a remarkable extent the process of Incubation: that curious persistence of heuristic tension through long periods of time during which the problem is not consciously entertained.

An extensive preoccupation with a problem imposes an emotional strain and a discovery which releases from it is a great joy. The story of Archimedes rushing out from his bath into the streets of Syracuse, shouting "Heureka!" is a witness to this; and the account I have quoted from Köhler of the way his chimpanzees behaved before and after solving a problem suggests that they also experience such emotions. I shall show this more definitely later. I mention it now only to make clear that nothing is a problem or discovery in itself; it can be a problem only if it puzzles and worries somebody, and a discovery only if it relieves somebody from the burden of a problem. A chess problem means nothing to a chimpanzee or to an imbecile and hence does not puzzle them; a great chess master on the other hand may fail to be puzzled by it because he finds its solution without effort; only a player whose ability is about equal to the problem will find intense preoccupation in it. Only such a player will appreciate its solution as a discovery.

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1. "The greatest impression on the visitor (writes Köhler) was made when Sultan made a pause, scratching his head leisurely and not moving anything but his eyes and very slightly his head, scrutinising the situation around him in the minutest detail." The Mentality of Apes, London 1927, p.200.

2. G. Wallas, The Art of Thought, London 1946, p.40 ff.

3. An ape which for a while had been searching for a tool to rake in a bunch of bananas lying outside its cage, and had made various fruitless attempts in this direction - such as trying to break off a board from the lid of a wooden case or hitting out with a stalk of straw in the direction of the prize - had apparently abandoned the task altogether. It went on playing with one of its fellows for about 10 minutes without turning again to the bananas outside the cage. Then suddenly, its attention having been diverted from its game by a shout nearby, its eyes happened to fall on a stick attached to the roof of the cage and at once it went for the stick and by jumping up a number of times finally secured it and hauled in the bananas by its aid. We may take this to show that even while otherwise occupied the animal kept its problem alive "at the back of its mind", keeping it ready to pounce on the instruments of a solution when they happened to meet its eye. Köhler, op.cit. p.184.

It appears possible to appraise the comparative hardness of a problem and to test the intelligence of subjects by their capacity for solving problems of a certain degree of hardness. The intelligence of chimpanzees and the hardness of certain problems were both successfully assessed by Köhler when he devised a series of problems which some of his apes could solve with some effort while others among them usually failed altogether to do so. The success of Yerkes in setting problems to earthworms (which these could solve after about a hundred trials), shows that he could assess even such extremely low powers of intelligence as were required here from the earthworm.[1] Editors of a crossword column undertake a similar feat in supplying their readers with a steady stream of always equally difficult problems. We may conclude that while a problem must always be regarded as being a problem to some kind of person, it is possible for an observer reliably to recognise it as such in respect to identifiable persons.

If an animal who has solved a problem is placed once more in the original situation, it proceeds unhesitatingly to apply the solution which it had originally discovered at the cost of much effort and perhaps many unsuccessful trials. This shows that by solving the problem the animal has acquired a new intellectual power which prevents it from being over again puzzled by the problem. Instead, it can now deal with the situation in a routine manner involving no heuristic tension and achieving no discovery. The problem has ceased to exist for it.

This irreversible character of heuristic acts is important. It suggests that no solution of a problem can be accredited as a discovery if it is achieved by a procedure following definite rules. For such a procedure would be reversible in the sense that it could be traced back stepwise to its beginning and repeated once more any number of times, like any arithmetical computation. Accordingly, any strictly formalised procedure would also be excluded as a means of achieving discovery.

It would follow that true discovery is not a strictly logical performance. Accordingly, we may describe the obstacle to be overcome in solving a problem as a 'logical gap', and speak of the width of the logical gap as the measure of the ingenuity required for solving the problem. 'Illumination' is then the leap by which the logical gap is crossed. It is the plunge by which we gain a foothold at another shore of reality. On such plunges the scientist has to stake bit by bit his entire professional life.

The width of the logical gap crossed by an inventor is subject to legal assessment. Courts of law are called upon to decide whether the ingenuity displayed in a suggested technical improvement is high enough to warrant its legal recognition as an invention or is merely a routine improvement, achieved by the application of known rules of the art. The invention must be acknowledged to be unpredictable, a quality which is assessed by the intensity of the surprise it might reasonably have aroused. This unexpectedness corresponds precisely to the presence of a logical gap between the antecedent knowledge from which the inventor started and the consequent discovery at which he arrived.

Established rules of inference offer public paths for drawing intelligent conclusions from existing knowledge. The pioneer mind which reaches its own distinctive conclusions by a leap across a logical gap deviates from the commonly accepted process of reasoning to achieve surprising results. Such an act is original in the sense of making a new start, and the capacity for initiating it is the gift of originality; a gift possessed by a small minority.

Since the Romantic movement originality has become increasingly recognised as a native endowment which alone enables a person to initiate an essential innovation. Universities and industrial research laboratories are founded today on the employment of persons with original minds. Permanent appointments are given to young scientists who are credited with signs of originality, in the expectation that they will continue to produce surprising ideas for the rest of their lives.

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1. R. M. Yerkes, "The Intelligence of Earthworms", Jour. Anim. Behav., Vol. II (1912), pp.332-352. cf. N. R. F. Maier and T. Schneirla, Principles of Animal Psychology, New York and London, 1935, pp.98-101.

Admittedly, there are minor heuristic acts within the power of ordinary intelligence and indeed continuous with the adaptive capacities of life down to its lowest levels. The interpretative framework of the educated mind is ever ready to meet somewhat novel experiences and to deal with them in a somewhat novel manner. In this sense all life is endowed with originality and originality of a higher order is but a magnified form of a universal biological adaptivity. But genius makes contact with reality on an exceptionally wide range: by seeing problems and reaching out to hidden possibilities for solving them, far beyond the anticipatory powers of current conceptions. Moreover, by deploying such powers in an exceptional measure - far surpassing our own as onlookers - the work of genius offers us a massive demonstration of a creativity which cannot be explained in other terms nor taken unquestioningly for granted. In confrontation with genius we are forced to acknowledge the originative power of life, which we might and commonly do overlook in its ubiquitous lesser manifestations; for by paying respect to another person's judgment as superior to our own, we emphatically acknowledge originality in the sense of a performance the procedure of which we cannot specify.

In choosing a problem the investigator takes a decision fraught with risks. The task may be insoluble or just too difficult. In that case his effort will be wasted and with it the effort of his collaborators, as well as the money spent on the whole project. But to play safe may be equally wasteful. Mediocre results are no adequate return for the employment of high gifts, and may not even repay the money spent on achieving them. So the choice of a problem must not only anticipate something that is hidden and yet not inaccessible but also assess the investigator's own ability (and those of his collaborators) against the anticipated hardness of the task, and make a reasonable guess as to whether the hoped for solution will be worth its price in terms of talent, labour and money. To form such estimates of the approximate feasibility of yet unknown prospective procedures leading to unknown prospective results is the day-to-day responsibility of anyone undertaking independent scientific or technical research. On such grounds as these he must even compare a number of different suggestions and select from them for attack the most promising problem. Yet experience shows that such a performance is possible and can even be relied upon with a considerable degree of probability.

2. Heuristic maxims.

There are three major fields of knowledge in which discoveries are possible: natural science, technology and mathematics. I have referred to examples from each of these fields to illustrate the anticipatory powers which guide discovery. These are clearly quite similar in all three cases. Yet the efforts of philosophers have been almost wholly concentrated on the process of empirical discovery which underlies the natural sciences. Ever since the rise of empiricism at the turn of the 16th century philosophers of science have been preoccupied with an attempt to define and justify the process of induction, while by contrast, nobody seems to have tried to define and justify the process by which technical innovations are made, as for example when a new machine is invented. The process of discovery in mathematics has received some attention, and has recently been attacked both from the logical and psychological point of view, but neither approach has raised any epistemological questions parallel to those so sedulously pursued for centuries in connection with empirical induction. It seems to me that any serious attempt to analyse the process of discovery should be sufficiently general to apply to all three fields of systematic knowledge and I should like to make here a possible contribution to this programme by identifying and acknowledging the powers on which we rely in solving mathematical problems. For reasons of space I shall exclude the history of major discoveries which often involve modifications in the foundations of mathematics and shall attend only to the type of problems that are set to students in teaching them mathematics. Since the solution of these problems is not known to the student the process of finding it bears the marks of a discovery, even though it involves no fundamental change of outlook.

The fact that the teaching of mathematics relies heavily on practice, shows that mathematical knowledge can be acquired only by developing an art; the art of solving mathematical problems. The same is true not only of mathematics and formal logic, but equally also of all mathematical sciences, like mechanics, electro-dynamics, thermodynamics and the mathematical branches of engineering; you cannot master any of these subjects without working out concrete problems in them. The art you strive for in such practical courses is that of converting a language, so far only receptively assimilated, into an effective tool for interpreting a new subject matter, which in this case consists in solving problems.

Thus the process by which mathematics is taught shows once more that the solving of mathematical problems is a heuristic act which leaps across a logical gap. While we cannot expect to find any strict rules for performing such an act, we may expect to discover certain rules of art, the interpretation of which is itself a part of the very art for the pursuit of which they offer us guidance. This is confirmed by the fact that the maxims of problem-solving can themselves be learnt only by practice. It is indeed above all the art of heuristic reasoning that the practical teaching of mathematics seeks to impart. This seems to me clearly proven by the comprehensive studies of G. Polya on the subject of mathematical heuristics on which I shall lean heavily for this study. [1]

The simplest heuristic effort is to search for an object you have mislaid. When I am looking for my fountain pen I know what I expect to find; I can name it and describe it. Though I know much more about my fountain pen than I can ever recall, and do not know exactly where I left it, the pen is clearly known to me and I know also that it is somewhere within a certain region, though I do not know where. My knowledge of the thing I am looking for is much less ample when I am looking for a word to fit into a crossword puzzle. This time I know only that the missing word has a certain number of letters and designates, for example, something that is badly needed in the Sahara or flows out of a central chimney. These properties are merely clues to a word that I definitely do not know; clues from which I must try to gain an intimation of what the unknown word may be. Again, a name which I know well but cannot recall at the moment lies somewhere halfway between those two cases. It is more closely present to my mind than the unknown solution of a crossword puzzle, but less closely perhaps than the mislaid fountain pen and its unknown location. Mathematical problems are in the class of crossword puzzles, for to solve such a problem we must find (or construct) something that we have never seen before, with the given data serving us as clues to it.

A problem may admit of a systematic solution. By ransacking my flat inch by inch I may make sure of eventually finding my fountain pen which I know to be somewhere in it. I might solve a chess problem by trying out mechanically all combination of possible moves and countermoves. Systematic methods apply also to many mathematical problems, though usually they are far too laborious to be carried out in practice.[2] It is clear that any such systematic operation would reach a solution without crossing a logical gap and would not constitute a heuristic act.

The difference between the two kinds of problem solving, the systematic and the heuristic, reappears in the fact, that while a systematic operation is a wholly deliberate act, a heuristic process is a combination of active and passive stages. A deliberate heuristic activity is performed during the stage of Preparation. If this is followed by a period of Incubation, nothing is done and nothing happens on the level of consciousness for this time. The advent of a bright idea (whether following immediately from Preparation or only after an interval of Incubation) is the fruit of the investigator's earlier efforts, but not itself an action on his part; it just happens to him. And again, the testing of the 'bright idea' by a formal process of Verification, is another deliberate action of the investigator. However, the decisive act of discovery must have occurred before this, at the moment when the happy thought emerged.

Though the solution of a problem is something we have never met before, yet in the heuristic process it plays a part similar to the mislaid fountain pen or the forgotten name which we know quite well. We are looking for it as if it were there, pre-existent. Problems set to students are of course known to have a solution; but the belief that there exists a hidden solution which we may be able to find, is essential also in envisaging and working at a yet unsolved problem. It determines also the manner in which the 'happy thought' eventually presents itself as something inherently satisfying. It is not one among a great many ideas to be pondered upon at

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1. G. Polya, How to Solve it, Princeton, 1945, and Mathematics and Plausible Reasoning, 2 Vols. London, 1954.
   Penetrating observations on problem solving have also been contributed by psychologists, mainly Duncker and Wertheimer.

2. A. M. Turing (Science News 31, 1954) has computed the number of arrangements that would have to be surveyed in the process of solving systematically a very common form of puzzle consisting of sliding squares to be rearranged in a particular way. The number is 20,922,789,888,000. Working continuously day and night and inspecting one position per minute the process would take 4 million years.

leisure, but one which carries conviction from the start. We shall see in a moment that this is a necessary consequence of the way a heuristic striving evokes its own consummation. To the close analysis of this process I shall now turn.

A problem is an intellectual desire (a 'quasi-need' in K. Lewin's terminology) and like every desire it postulates the existence of something that can satisfy it; in the case of a problem its satisfier is its solution. As all desire stimulates the imagination to dwell on the means of satisfying it, and is stirred up in its turn by the play of the imagination it has fostered, so also by taking interest in a problem we start speculating about its possible solution and in doing so become further engrossed in the problem.

Obsession with one's problem is in fact the mainspring of all inventive power. Asked by his pupils in jest what they should do to become 'a Pavlov', the master answered in all seriousness: "get up in the morning with your problem before you. Breakfast with it. Go to the laboratory with it. Eat your lunch with it. Keep it before you after dinner. Go to bed with it in your mind. Dream about it."[1] It is the unremitting preoccupation with a problem that lends to genius its proverbial capacity for taking infinite pains. And the intensity of our preoccupation with a problem generates also our power for reorganising our thoughts successfully, both during the hours of search and afterwards, during a period of rest.[2]

But what is the object of this intensive preoccupation? How can we concentrate our attention on something we don't know? Yet this is precisely what we are told to do: "Look at the unknown!" - says Polya " "Look at the end. Remember your aim. Do not lose sight of what is required. Keep in mind what you are working for. *Look at the unknown*. *Look at the conclusion*."[3] No advice could be more emphatic.

The seeming paradox is resolved by the fact that even though we have never met the solution we have a conception of it in the same sense as we have a conception of a forgotten name. By directing our attention on a focus in which we are subsidiarily aware of all the particulars that remind us of the forgotten name, we form a conception of it; and likewise, by fixing our attention on a focus in which we are subsidiarily aware of the data by which the solution of a problem is determined, we form a conception of this solution. The admonition to look at the unknown really means that we should *look at the known data, not, however, in themselves, but as clues to the unknown; as pointers to it and parts of it.* We should make every effort to feel our way to an understanding of the manner in which these known particulars hang together both mutually and with the unknown. Thus we must make sure in the first place that the unknown is really there, essentially determined by what is known about it, and able to satisfy all the demands made on it by the problem.

All our conceptions have heuristic powers; they are ever ready to identify novel instances of experience by modifying themselves so as to comprise them. The practice of skills likewise is inventive; by concentrating our purpose on the achievement of success we evoke ever new capacities in ourselves. A problem partakes of both these types of endeavour. It is a conception of something we are striving for. It is an intellectual desire for crossing a logical gap on the other side of which lies the unknown: fully marked out by our conception of it, though as yet never seen in itself. The search for a solution consists in casting about with this purpose in mind. This we do by performing two operations which must always be tried jointly. We must (1) set out the problem in suitable symbols and continuously reorganise its representation with a view to eliciting some new suggestive aspects of it and concurrently (2) ransack our memory for any similar problem of which the solution is known. The scope of these two operations will usually be limited by the student's technical facility for transforming the given data in different ways and by the range of germane theorems with which he is acquainted. But his success will depend ultimately on his capacity

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1. J. R. Baker, *Science and the Planned State*, London, 1945, p.35.

2. "Only such problems come back improved after a rest whose solution we passionately desire and for which we have worked with great tension" writes Polya (op. cit., p.172).

3. *ibid*, p.112. italics in the original.

for sensing the presence of yet unrevealed logical relations between the conditions of the problem, the theorems known to him, and the unknown solution he is looking for. Unless his casting about is guided by a reliable sense of growing proximity to the solution, he will make no progress towards it. Conjectures made at random, even though following the best rules of heuristics, would be hopelessly inept and totally fruitless.

The process of solving a mathematical problem continues to depend therefore at every stage on the same ability to anticipate a hidden potentiality which enables the student to see a problem in the first place and set out to solve it. Polya has compared a mathematical discovery consisting of a whole chain of consecutive steps with an arch where every stone depends for its stability on the presence of others, and pointed out the paradox that the stones are in fact put in one at a time. Again the paradox is resolved by the fact that each successive step of the incomplete solution is upheld by the heuristic anticipation which originally evoked its invention: by the feeling that its emergency has narrowed further the logical gap of the problem.

The growing sense of approaching to the solution of a problem can be commonly experienced when we grope for a forgotten name. We all know the exciting sense of increasing proximity to the missing word which we may confidently express by saying: "I shall remember it in a moment" and perhaps later "It is on the tip of my tongue." The expectation expressed by such words is often confirmed in the event. I believe that we should likewise acknowledge our capacity both to sense the accessibility of a hidden inference from given premises and to invent transformations of the premises which increase the accessibility of the hidden inference. We should recognise that this foreknowledge biasses our guesses in the right direction, so that their probability of hitting the mark, which would otherwise be zero, becomes so high that we can definitely rely on it simply on the grounds of a student's intelligence: or for higher performances, on the grounds of the special gifts possessed by the professional mathematician.

The feeling that the logical gap separating us from the solution of a problem has been reduced means that less work should remain to be done for solving it. It may also mean that the rest of the solution will be comparatively easy or that it may present itself without further effort on our part, after a period of rest. The fact that our intellectual strivings make effective progress during a period of Incubation without any effort on our part is in line with the latent character of all knowledge. As we continuously know a great many things without always thinking of them, so we naturally also keep on desiring or fearing all manner of things without always thinking of them. We know how a set purpose may result in action automatically later, as when we go to bed resolved to wake up at a certain hour. Post-hypnotic suggestions can set going latent processes which compulsively result after a number of hours in the performance requested of the subject.[1] Mrs. Zeigarnik has shown that unfinished tasks continue likewise to preoccupy us unconsciously; their memory persists after finished tasks are forgotten.[2] The fact that the tension set up by the unfinished task continues to make progress towards its fulfilment, is shown by the well known experience of sportsmen that a period of rest following on a spell of intensive training produces an improvement of skill. The spontaneous success of the search for a forgotten name or for the solution of a problem, after a period of quiescence, falls in line with this experience.

These antecedents explain also the manner in which the final success of problem solving will suddenly set in. For each step, whether spontaneous or contrived, that brings us nearer to the solution increases our premonition of its proximity and brings a more concentrated effort to bear on a reduced logical gap. The last stage of the solution may therefore be frequently achieved in a self-accelerating manner and the final discovery may be upon us in a flash.

1. Cf. N. Ach, "Determining Tendencies; Awareness;" in D. Rappaport, Organisation and Pathology of Thought, New York, 1951, p.16 ff.

2. W. D. Ellis, A Source Book of Gestalt Psychology, New York, 1938, p.300-314.

I have said that our heuristic cravings imply, like our bodily appetities, the existance of something which has the properties required to satisfy us, and that the intimations which guide our striving express this belief. But the satisfier of our craving has in this case no bodily existence; it is not a hidden object but an idea never yet conceived. We hope that as we work at the problem this idea will come to us, whether all at once or bit by bit; and only if we believe that this solution exists can be passionately search for it and evoke from ourselves heuristic steps towards its discovery. Therefore as it emerges in response to our search for something we believe to be here, discovery, or supposed discovery, will always come to us with the conviction of its being true. It arrives accredited in advance by the heuristic craving which evoked it.

The most daring feats of originality are still subject to this law: they must be performed in the assumption that they originate nothing, but merely reveal what is there. And their triumph confirms this assumption, for what has been found bears the mark of reality in being pregnant with yet unforeseeable implications. Mathematical heuristics, aiming at conceptual reorganisation without reference to new experience, once more exemplifies in its own terms that an intellectual striving entails its conviction of anticipating reality. It illustrates also how this conviction finds itself confirmed by the eventual solution, which "solves" precisely because it successfully claims to reveal an aspect of reality. And we can see again, finally, that this whole process of discovery and confirmation ultimately relies on our own accrediting of our own vision of reality.

To start working on a mathematical problem, we reach for pencil and paper, and throughout the stage of Preparation we keep trying out ideas on paper in terms of symbolic operations. If this does not lead straight to success, we may have to think the whole matter over again, and may perhaps see the solution revealed much later unexpectedly in a moment of Illumination. Actually, however, such a flash of triumph usually offers no solution, but only envisages a solution, which has yet to be tested. In the verification or working out of the solution we must again rely therefore on explicit symbolic operations. Thus both the first active steps undertaken to solve a problem and the final garnering of the solution rely effectively on computations and other symbolic operations, while the more informal act by which the logical gap is crossed lies between these two formal procedures. However, the intuitive powers of the investigator are always dominant and decisive. Good mathematicians are usually found capable of carrying out computations quickly and reliably, for unless they command this technique they may fail to make their ingenuity effective; but their ingenuity itself lies in producing ideas. Hadamard says that he used to make more mistakes in calculations than his own pupils but that he more quickly discovered them because the result did not <u>look</u> right; it is almost as if by his computations he had been merely drawing a portrait of his conceptually prefigured conclusions.[1] Gauss is widely quoted as having said: "I have had my solutions for a long time but I do not yet know how I am to arrive at them." Though the quotation may be doubtful it remains well said.[2] A situation of this kind certainly prevails every time we discover what we believe to be the solution to a problem. At this moment we have the vision of a solution which <u>looks</u> right and which we are therefore confident to <u>prove</u> right.[3]

1. Hadamard, <u>The Psychology of Convention in the Mathematical Field</u>, Princeton, 1945, p.49.

2. Agnes Arber, <u>The Mind and the Eye</u>, Cambridge, 1954, p.47.

3. Archimedes describes in his 'Method' a mechanical process of geometrical demonstration which carries conviction with him, though he regards its results as still requiring proof, which he proceeds to supply. B. L. Van der Waerden, <u>Science Awakening</u>, Groningen, 1954, p.215.

The manner in which the mathematician works his way towards discovery by shifting his confidence from intuition to computation and back again from computation to intuition, while never releasing his hold on either of the two, represents in miniature the whole range of operations by which articulation disciplines and expands the reasoning powers of man. This alternation is asymmetrical, for a formal step can be valid only by virtue of our tacit confirmation of it. Moreover, a symbolic formalism is itself but an embodiment of our antecedent unformalized powers; it is an instrument skilfully contrived by our inarticulate selves for the purpose of relying on it as our external guide. The interpretation of primitive terms and axioms is predominantly inarticulate and so is the process of their expansion and re-interpretation which underlies the progress of mathematics. A formal proof proves nothing until it induces the tacit conviction that it is binding. Thus the alternation between the intuitive and the formal depends on tacit affirmations both at the beginning and at the end of each chain of formal reasoning.